$$\sigma_2 = (1 - \varkappa n_1^2)\,\sigma_0^{11} + [2\nu\xi - \varkappa\,(1 - 2\nu) n_1^2]\sigma_0^{22} + [(\varkappa + 2\xi^{-1}) n_1^2 - 2\nu]\sigma_0^{33} - \tau^{13}$$
$$\tau = (1 + \xi)\,[(\varkappa\xi)^{-1}(1 + \xi) n_1\sigma_0^{23} - n_3\sigma_0^{12}]$$
$$\tau^{13} = 2\varkappa\,(1 + \xi^{-1}) n_1 n_3 \sigma_0^{13}$$

In the section $n_3 = 0$

$$\sigma_1 = (1 - \varkappa n_1^2)\sigma_0^{11} + [(\varkappa + 2\xi) n_1^2 - 2\nu]\sigma_0^{22} + [2\nu\xi^{-1} - \varkappa\,(1 - 2\nu)\, n_1^2]\sigma_0^{33} - \tau^{12}$$
$$\sigma_2 = -\nu\varkappa n_1^2\sigma_0^{11} + [\nu\,(\varkappa + 2\xi) n_1^2 - 1]\sigma_0^{22} + [1 + 2\xi^{-1} - \nu\varkappa\,(1 - 2\nu) n_1^2]\sigma_0^{33} - \nu\tau^{12}$$
$$\tau = \xi^{-1}\,(1 + \xi)\,[\varkappa^{-1}\,(1 + \xi) n_1\sigma_0^{23} - n_2\sigma_0^{13}]$$
$$\tau^{12} = 2\varkappa\,(1 + \xi) n_1 n_2 \sigma_0^{12},\ \varkappa = (1 - \nu)^{-1}$$

Therefore, explicit expressions in terms of the ellipsoid parameters, the elastic constants of the external medium, and the coordinates of the unit normal vector are obtained for the stresses on the needle surface. Analysis of these expressions shows that the anisotropy of the medium introduces both quantitative and qualitative changes into the behaviour of the stresses.

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A METHOD OF CONSTRUCTING WEIGHTING FUNCTIONS FOR A CIRCULAR CRACK*

A.N. BORODACHEV

A general formulation is used to consider a static problem for a linearly elastic body with internal circular crack of normal separation. It is shown that the corresponding weighting function enabling a direct calculation to be made of the stress intensity factor (SIF) under arbitrary loading conditions is equal to the product of the axisymmetric weighting function and Poisson's kernel. The known axisymmetric solution /1/ is used to construct, as an example, the weighting function for a circular crack in an unbounded inhomogeneous body with periodic law of variation in the value of Poisson's ratio. An asymptotic analysis of the solution obtained is carried out for a material with rapidly oscillating elastic characteristics.

Some problems of the inhomogeneous theory of elasticity were studied for bodies with variable Poisson's ratio in /1-4/.

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1. ***Initial relationships and basic notation.*** Let a linearly elastic body occupy the region Ω of three-dimensional Euclidean space $\mathbf{R}^3$, bouned by the surface $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ where $\partial\Omega_1$ and $\partial\Omega_2$ have no common internal points. There are no volume forces, the homogeneous kinematic forces are specified on $\partial\Omega_1$ and homogeneous static boundary conditons on $\partial\Omega_2$. The body has an internal circular crack of radius a, whose surfaces are described in a cylindrical system of coordinates by the relations $S\pm = \{r, \theta, z\colon 0 \leqslant r \leqslant a,\ -\pi \leqslant \theta < \pi,\ z = 0\pm\}$.

We shall restrict ourselves to the case (the most important one from the practical point of view) of normal separation when no shear stresses exist at the points of the plane $z = 0$ belonging to Ω, and the crack surfaces are acted upon by a selfequilibrated system of normal loads $\sigma_z(r, \theta, 0\pm) = -p(r, \theta),\ r \leqslant a$.

Other possible formulations of the problem for a body with a crack of normal separation can be reduced to the present formulation using the superposition principle /5/.

We shall assume that the problem belongs to the class $\mathbf{A}$, provided that the geometry of the region Ω, the partition of the boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and other properties of the material (i.e. the character of possible inhomogeneities and the anisotropy) are such, that, when the crack surfaces are loaded axisymmetrically, then its solution is independent of the angular coordinate.

Let two unit, concentrated normal forces acting in opposite directions (cleaving forces) be applied to the crack surface at the points with coordinates $(r, \theta, 0\pm)$. The value of the stress intensity factor (SIF) of normal separation corresponding to such loading and called the weighting function /5/, will be denoted by $k(r, \theta; \varphi)$ where φ is the value of the angular coordinate for the point in question lying on the crack contour. For the problems of class $\mathbf{A}$, it is clear that $k(r, \theta; \varphi) \equiv k(r, \theta - \varphi)$ where $k(r, \theta)$ is an even, 2π-periodic function of the angular coordinate and $k(r, \theta) > 0$ as $r < a$.

In the case of arbitrary loading of the crack surfaces the value of the SIF can be found directly from the weighting function with help of the quadratures

$$K_1(\varphi) = \int_0^a \int k(r, \theta - \varphi)\, p(r, \theta)\, r\, d\theta\, dr \tag{1.1}$$

(the integration over θ and φ is carried out everywhere over the segment $[-\pi, \pi]$).

In the case of problems of class $\mathbf{A}$ we introduce, apart from the weighting function, the axisymmetric weighting function $k(r')$, equal to the value of the SIF when the crack surfaces are loaded according to the law $p(r, \theta) = (2\pi r)^{-1}\delta(r - r')$ where $\delta(r)$ is the Dirac function. Substituting this load into (1.1) and taking into account the 2π-periodicity of the function $k(r, \theta)$ we obtain the following relation between the weighting function and the axisymmetric weighting function:

$$k(r) = (2\pi)^{-1} \int k(r, \theta) d\theta,\ r \leqslant a \tag{1.2}$$

The axisymmetric weighting function has an (integrable) singularity at the points of the boundary crack contour; therefore relation (1.2) should be regarded, when $r = a$, as the existence of the limit

$$\lim_{r \to a-0} \int \frac{k(r, \theta)\, d\theta}{2\pi k(r)} = 1 \tag{1.3}$$

Another representation for the axisymmetric weighting function can be obtained using the variational formula for a body with a normal separation crack /6-8/

$$\delta_n u_z(r, \theta, 0^+) = aC \int k(r, \theta - \varphi) K_1(\varphi) \delta n(\varphi) d\varphi \tag{1.4}$$

which determines the variation in the normal displacements of the points on the crack surface S^+ caused by variation $\delta n(\varphi)$ in the boundary contour of the crack. The displacements $u_z(r, \theta, 0^+)$ and SIF $K_1(\varphi)$ in (1.4) correspond to some (arbitrary) law of loading $p(r, \theta)$.

The constant C from (1.4) depends only on the elastic properties of the material and is calculated from the principal terms of the asymptotic expansions of the normal stresses and displacements near the crack contour. Thus if

$$\sigma_z = K_1 (2\rho)^{-1/2},\ u_z = cK_1 (2\rho)^{1/2},\ \rho \to 0 \tag{1.5}$$

where ρ is the distance to the crack contour in the plane $z = 0$, then $C = \pi c/2$ /7/.

Suppose that for some axisymmetric loading law $p(r, \theta) = p(r)$ when, for problems of class $\mathbf{A}$ $u_z(r, \theta, 0^+) = u_z(r, 0^+)$ and $K_1(\varphi) = K_1 \equiv \text{const}$, the crack contour obtains an axisymmetric variation $\delta n(\varphi) = \delta a$. The resulting variation of the normal displacements of the points on the crack surface S^+ is calculated as follows:

$$\delta_n u_z(r, 0^+) = (\partial u_z(r, 0^+)/\partial a)\delta a$$

and the variational formula (1.4) will take the form

$$(aCK_1)^{-1}\partial u_z(r, 0^+)/\partial a = \int k(r, \theta)d\theta, \ r \leqslant a \tag{1.6}$$

where the properties of evenness and 2π-periodicity of the function $k(r, \theta)$ with respect to the angular coordinate, are taken into account. It is clear that $\partial u_z(r, 0^+)/\partial a$ has a singularity at the points of the boundary contour of the crack; therefore this relation should be regarded, when $r = a$, in the same manner as (1.2).

Comparing relations (1.2) and (1.6) we obtain the following result:

$$k(r) = (\pi^2 acK_1)^{-1}\partial u_z(r, 0^+)/\partial a \tag{1.7}$$

Thus in order to determine the axisymmetric weighting function it is sufficient to know the solution of the problem corresponding to any axisymmetry law of loading the surfaces of the crack $p(r)$. The invariance of (1.7), relative to the axisymmetric loading laws $p(r)$, is governed, obviously, by the invariance of the variational formula (1.4) with respect to more-general loading laws $p(r, \theta)$. We note that, as it applies to homogeneous and isotropic materials, relation (1.7) follows from the results of /9/.

2. ***The general structure of weighting functions for a circular crack.*** By virtue of relation (1.1) the axisymmetric weighting function $k(r)$ can be calculated directly from the weighting function $k(r, \theta)$. We shall show that the inverse relation, which is more important, also holds.

Indeed, taking into account relations (1.2) and (1.3) we see that the formula

$$1 = (2\pi)^{-1}\int [k(r, \theta)/k(r)]\, d\theta \tag{2.1}$$

defines the representation of a function equal to unity within, and at the boundary of a circle of radius a. The function, which is harmonic within the circle, is defined uniquely in terms of its boundary conditions.

We know /10/ that a function harmonic within a circle and taking unit values at its boundary, has the form

$$1 = (2\pi)^{-1}\int P_r(\theta)d\theta \tag{2.2}$$

where the Poisson's kernel is given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty}\left(\frac{r}{a}\right)^{|n|} e^{in\theta} \equiv \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos\theta}$$

Comparing the representations of the functions (2.1) and (2.2) harmonic in the circle, we conclude that the general structure of the weighting function is established by the relation

$$k(r, \theta - \varphi) = k(r)P_r(\theta - \varphi) \tag{2.3}$$

and this allows us to formulate the main result of this paper.

Theorem. For problems of class A the weighting function is equal to the product of the axisymmetric weighting function and a Poisson's kernel.

Generally speaking, the weighting function is determined uniquely from any axisymmetric solution using relations (2.3) and (1.7).

In particular, from representation (2.3) it follows that $k(r, \theta)\,[2\pi k(r)]^{-1}$ represents a so-called approximate unity whose properties have been described in /10/.

The structure of the weighting function for the problems of class A enables us to transform formula (1.1) to a form more suitable for computations. Indeed, with few practical exceptions, the load $p(r, \theta)$ can be represented by a complex Fourier series

$$p(r, \theta) = \sum_{n=-\infty}^{\infty} p_n(r)\, e^{in\theta}, \quad p_n(r) = (2\pi)^{-1}\int p(r, \theta)\, e^{-in\theta}\, d\theta$$

Substituting this series, together with (2.3), into (1.1) and integrating over the angular coordinate, we arrive at the following expression for the SIF in which the dependence on the polar angle is given explicitly

$$K_1(\varphi) = 2\pi \sum_{n=-\infty}^{\infty} \frac{e^{in\varphi}}{a^{|n|}} \int_0^a p_n(r)\, k(r)\, r^{|n|+1}\, dr \tag{2.4}$$

Thus the value of the SIF is given, in the case of an arbitrary law of loading the crack surfaces $p(r, \theta)$, in terms of the axisymmetric weighting function $k(r)$ by the relation (2.4).

We shall use, as an illustrative example, the problem of a circular crack in an unbounded ($\Omega = \mathbf{R}^3$), homogeneous and isotropic elastic body. The boundary conditions on $\partial\Omega$ are replaced here by the usual conditions of decay at infinity.

The simplest solution of the problem in question corresponding to the loading law $p(r) = p \equiv \mathrm{const}$, has the form /11/

$$u_z(r, 0^+) = \frac{2p(1-\nu)}{\pi\mu}(a^2 - r^2)^{1/2}, \quad r \leqslant a \tag{2.5}$$

$$u_z = \frac{1-\nu}{\mu} K_1 (2\rho)^{1/2}, \quad \rho \to 0; \quad K_1 = \frac{2pa^{1/2}}{\pi}$$

where ν is Poisson's ratio and μ is the shear modulus.

Calculating the axisymmetric weighting function directly from formula (1.7), we obtain, using (2.5),

$$k^0(r) = [\pi^2 a^{1/2} (a^2 - r^2)^{1/2}]^{-1} \tag{2.6}$$

and this agrees with the known result in /12/.

Substituting (2.6) into (2.3) we also obtain the well-known result /12/

$$k^0(r, \theta - \varphi) = \frac{\pi^{-2} a^{-1/2} (a^2 - r^2)^{1/2}}{a^2 + r^2 - 2ar\cos(\theta - \varphi)} \tag{2.7}$$

whose derivation by traditional methods based on the solution of the non-axisymmetric boundary-value problem involves copious computations.

3. ***The weighting function for a circular crack in an inhomogeneous body.*** The proposed method enables us to extend appreciably the range of application of the weighting functions for a circular crack, since the corresponding axisymmetric solutions (and, in particular, the axisymmetric weighting functions), are known for a number of problems /11-13/. The use of approximate (or asymptotic) axisymmetric solutions will lead, naturally, to approximate (asymptotic) representations for the weighting function.

Below we shall consider the static, axisymmetric problem of a circular crack of normal separation, situated in an unbounded ($\Omega = \mathbf{R}^3$) inhomogeneous elastic body. The shear modulus of the material is constant ($\mu = \mathrm{const} > 0$) and Poisson's ratio $\nu(z)$ is a continuous (or piecewise-continuous) even function of the z coordinate, satisfying the standard conditions: $-1 < \nu(z) \leqslant 1/2$. In this case the modulus of elasticity of the material $E(z) = 2\mu\,[1 + \nu(z)]$ will be a positive function of the distance to the crack plane.

We note that the lack of analytic solutions of the problems of cracks in bodies with continuously variable Poisson's ratio was pointed out in /14/.

The problem formulated here belongs to class A and its axisymmetric solution has the form /1/

$$u_z(r, 0^-) = \frac{2(1-\nu_0)}{\pi\mu} \int_r^a \frac{h(x)\, dx}{(x^2 - r^2)^{1/2}}, \quad r \leqslant a \tag{3.1}$$

$$u_z = \frac{1-\nu_0}{\mu} K_1 (2\rho)^{1/2}, \quad \rho \to 0; \quad K_1 = \frac{2h(a)}{\pi a^{1/2}}$$

where $\nu_0 = \nu(0)$, and $h(x)$ is a solution of the integral Fredholm equation of the second kind with a symmetric kernel

$$h(x) + \int_0^a K(x, s)\, h(s)\, ds = \int_0^x \frac{p(r)\, r\, dr}{(x^2 - r^2)^{1/2}}, \quad 0 \leqslant x \leqslant a \tag{3.2}$$

$$K(x, s) = \frac{2}{\pi} \int_0^\infty G(t) \sin(xt) \sin(st)\, dt$$

$$1 + G(t) = 2(1 - \nu_0)\, t L(t), \quad \lim_{t \to \infty} G(t) = 0$$

$$L(t)=\int_0^\infty \gamma(z)e^{-2tz}\,dz,\quad \gamma(z)=[1-\nu(z)]^{-1}$$

Having calculated the axisymmetric weighting function with the help of formula (1.7) and taking into account relations (3.1), we obtain

$$k(r)=k^\circ(r)[1+\Lambda(r)] \tag{3.3}$$

$$\Lambda(r)=(a^2-r^2)^{1/2}\int_r^a \frac{H(x)\,dx}{(x^2-r^2)^{1/2}},\quad H(x)=\frac{1}{h(a)}\cdot\frac{\partial h(x)}{\partial a}$$

and hence

$$k(r,\theta-\varphi)=k^\circ(r,\theta-\varphi)[1+\Lambda(r)] \tag{3.4}$$

where the weighting functions $k^\circ(r)$ and $k^\circ(r,\theta)$ are given by relations (2.6) and (2.7).

The effect of the inhomogeneity of elastic material on the magnitude of the weighting function $k(r,\theta)$ from (3.4) is fully described by the function $\Lambda(r)$, since $k^\circ(r,\theta)$ is independent of the elastic characteristics of the material.

By virtue of relations (3.3), the construction of the function $\Lambda(r)$ reduces, essentially, to determining the function $H(x)$ by calculating the derivative with respect to a of the function $h(x)$, which is a solution of the Fredholm integral Eq.(3.2) for some axisymmetric loading law $p(r)$. In most cases numerical methods must be used to determine $h(x)$; therefore the determination of the derivative mentioned above may present considerable difficulties.

Note that the function $H(x)$ can be obtained without a preliminary determination of the function $h(x)$. Indeed, differentiating with respect to a the integral Eq.(3.2) we find, that $H(x)$ is a solution of the Fredholm integral equation

$$H(x)+\int_0^a K(x,s)H(s)\,ds=-K(x,a),\quad 0\leqslant x\leqslant a \tag{3.5}$$

whose right-hand side is independent of the loading law $p(r)$ and the kernel is given by relations (3.2).

Exact analytic solutions of the integral Eqs.(3.2) and (3.5) can be obtained within the framework of the model under discussion, for certain specific laws of inhomogeneity of the elastic material. For example, suppose

$$\gamma(z)\equiv[1-\nu(z)]^{-1}=b_1+b_2\cos b_3z \tag{3.6}$$

when $\nu(z)$ and $E(z)$ are periodic even functions of the z coordinate (more general periodic laws of inhomogeneity allowing the construction of analytic solutions are given in /4/). We shall assume that $b_3\geqslant 0$, which by virtue of the fact that the function $\cos x$ is even, does not lead to loss of generality.

The solution of integral Eq.(3.2) with an arbitrary right-hand side constructed in /1/ for the law of inhomogeneity (3.6) assumes, in connection with (3.5), the form

$$H(x)=b\eta B\,\mathrm{sh}(\eta x)[\mathrm{sh}(\eta a)+B\,\mathrm{ch}(\eta a)]^{-1} \tag{3.7}$$

$$b=b_2/b_1,\quad B=(1+b)^{-1/2},\quad \eta=b_3B/2$$

Substituting (3.7) into the second relation of (3.3), we obtain

$$\Lambda(r)=\frac{b\beta B(1-R^2)^{1/2}}{\mathrm{sh}\,\beta+B\,\mathrm{ch}\,\beta}\int_R^1\frac{\mathrm{sh}(\beta x)\,dx}{(x^2-R^2)^{1/2}} \tag{3.8}$$

$$R=r/a,\quad \beta=B\alpha,\quad \alpha=ab_3/2$$

and in particular

$$\Lambda(0)=b\beta B(\mathrm{sh}\,\beta+B\,\mathrm{ch}\,\beta)^{-1}\,\mathrm{shi}\,\beta \tag{3.9}$$

where $\mathrm{shi}\,\beta$ is the integral hyperbolic sine /15/.

Thus the function Λ characterizing the effect of the inhomogeneity of the elastic material on the magnitude of the SIF, depends on three dimensionless parameters b, α and R only.

We will establish the domains of admissible values of these parameters. It is clear that

$0 \leqslant \alpha < \infty$ and $0 \leqslant R \leqslant 1$, and the range of values of the parameter b is determined by the range of admissible values of the function $\nu(z)$. If we limit ourselves to the case when $0 \leqslant \nu(z) \leqslant {}^1/_2$ (we note that the existence of materials with negative Poisson's ratio was pointed out in /16/, then we can show /4/ that $-{}^1/_3 \leqslant b \leqslant {}^1/_3$.

When $\alpha = 0$ or when $b = 0$ (these values correspond to the homogeneous materials), $\Lambda(r) = 0$ and relation (3.4) becomes an identity. Moreover $\Lambda(r) \to 0$ as $R \to 1$, or, which is the same, as $r \to a$.

The study of the asymptotic behaviour of the function $\Lambda(r)$ as $\alpha \to \infty$ (which is equivalent, at a fixed value of the radius of the crack, to the case $b_3 \to \infty$), is of particular interest. In this case the functions $\nu(z)$ and $E(z)$ become rapidly oscillating and their period $T = 2\pi b_3^{-1} \to 0$.

Carrying out the passage to the limit in (3.8) and (3.9), we obtain

$$\lim_{\alpha\to\infty} \Lambda(r) = b\,[1 + (1 + b)^{1/2}]^{-1},\ 0 \leqslant R < 1 \tag{3.10}$$

so that for sufficiently large values of the parameter α the value of the function $\Lambda(r)$ will be determined in terms of the parameter b only, and will not depend on R.

Substituting the representation (3.3) into (2.4) and taking into account relation (3.10), we obtain

$$\lim_{\alpha\to\infty} K_1(\varphi) = (1 + b)^{1/2} K_1^{\circ}(\varphi) \tag{3.11}$$

where $K_1^{\circ}(\varphi)$ is the corresponding SIF for a circular crack in an unbounded homogeneous body, obtained by substituting relation (2.6) into (2.4). Therefore, for sufficiently large values of the parameter α the approximate value of the SIF can be found for an arbitrary loading law for the crack surfaces, directly from the formula $K_1(\varphi) \approx (1 + b)^{1/2} K_1^{\circ}(\varphi)$.

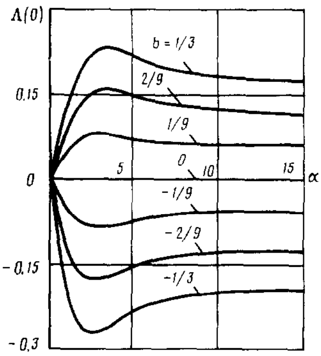

Fig.1

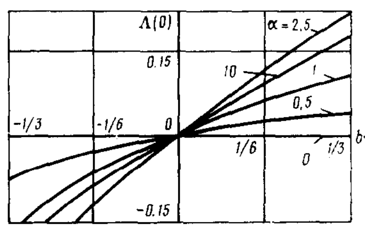

Fig.2

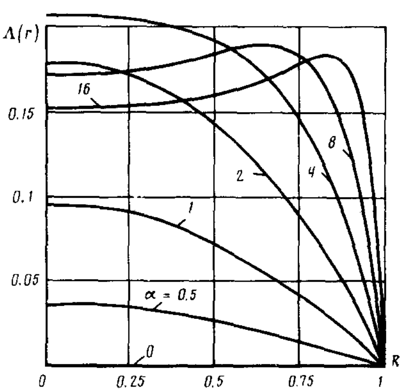

Fig.3

Fig.1 shows the relation between $\Lambda(0)$ and α for fixed values of b, and we see from them, in particular, that $|\Lambda(0)|$ reaches its maximum at $\alpha \approx 2.5$-3.5. When α is increased further the quantity $\Lambda(0)$ will tend monotonically to its limiting value given by the relation (3.10).

Fig.2 shows relations connecting $\Lambda(0)$ with b for fixed values of α, and we see that they are nearly linear.

The nature of the dependence of $\Lambda(r)$ on R for fixed values of α is shown in Fig.3 for the case of $b = 0.3$.

4. Integral characteristics of an inhomogeneous body with a circular crack. Since the quantity $K_1^\circ(\varphi)$ is independent of the elastic characteristics of the material, it follows that by virtue of relation (3.11) it is clear that when the SIF is being determined, the inhomogeneous material of the type in question, with rapidly oscillating values of $\nu(z)$ and $E(z)$, cannot be replaced by the equivalent (in the sense indicated above) homogeneous material with effective elastic characteristics. We shall show that such a replacement is possible when computing the integral characteristics of the problem which include, in particular, the magnitude of the volume of the crack and the potential energy of the body with a crack.

Using relations (3.1) and (3.2) we can show that the volume of a circular crack acted upon by a constant internal pressure ($p(r) = p \equiv \text{const}$) can be determined, for the law of inhomogeneity (3.6), from the formula

$$V = \frac{8(1-\nu_0)pa^3}{3\mu}\left\{1 + b - \frac{3b(1+\alpha)(\beta - \operatorname{th}\beta)}{\beta^3[1+(1+b)^{1/2}\operatorname{th}\beta]}\right\}$$

from which it follows, in particular, that

$$\lim_{\alpha\to\infty} V = 8(1-\nu_0)pa^3(1+b)(3\mu)^{-1} \tag{4.1}$$

Let us denote by $V^* = 8(1-\nu^*)pa^3(3\mu)^{-1}$ the volume of a circular crack situated within an unbounded homogeneous body with shear modulus μ and Poisson's ratio ν^*, acted upon by a constant internal pressure p. If we choose the quantity ν^* so that the following relation holds:

$$1 - \nu^* = (1-\nu_0)(1+b) \tag{4.2}$$

then relation (4.1) will be reduced to the following: $\lim_{\alpha\to\infty} V = V^*$. In other words, on changing, within the framework of the model in question, to materials with rapidly oscillating elastic characteristics $\nu(z)$ and $E(z)$ (more accurately when $\alpha \to \infty$), the value of the crack volume will tend to the corresponding value for a homogeneous material with Poisson's ratio satisfying relation (4.2).

The physical meaning of this relation is clarified when the elastic constant $\gamma^* = (1-\nu^*)^{-1}$ is introduced. This constant can be expressed, by virtue of the relations (3.6) and (4.2), in terms of the mean value of the function $\gamma(z) = [1-\nu(z)]^{-1}$ over the period

$$\gamma^* = \langle\gamma(z)\rangle \equiv \frac{1}{T}\int_0^T \gamma(z)\,dz \tag{4.3}$$

and from this we see, in particular, that $0 \leqslant \nu^* \leqslant 1/2$ when $0 \leqslant \nu(z) \leqslant 1/2$.

Generally speaking, the above asymptotic behaviour of the value of crack size occurs not only under the action of constant internal pressure, but also for any other type of loading of the crack surfaces $p(r, \theta)$. This can be shown using the relations which allow us to express the crack size for arbitrary loading of its surfaces, by the solution of the problem of a crack acted upon by a constant internal pressure /17/.

As we know /17/, the potential energy W of the body with a crack under the action of a constant internal pressure is connected with the crack volume by the following simple relation: $W = 1/2 pV$. Consequently, when $\alpha \to \infty$, relation $W \to W^*$ holds. Here W^* is the corresponding value of potential energy in the case of a homogeneous material with shear modulus μ and elastic constant γ^* equal, by virtue of (4.3), to the mean value of the function $\gamma(z)$.

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ON THE DEFORMATION OF AN ELASTIC HALF-SPACE WITH A THIN SLIT FOR MIXED CONDITIONS ON ITS BOUNDARY*

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The problem of the state of stress and strain in an elastic half-space with a cutout in the shape of a circular slot is solved by Kelvin's method /1/. The conditions on the slot are satisfied in a suitable manner by selecting the scalar and vector mass force potentials as generalized functions concentrated at the slot. The problem reduces to a system of Fredholm integral equations of the second kind in a semi-infinite interval. The solution for an elastic space with a slot is obtained in final form in the limiting case, which enables an estimate to be made of the magnitude of the settling of the earth's surface as a result of oil or gas deposit development.

1. Formulation of the problem. The axisymmetric problem of the stress and strain distribution in an elastic half-space E containing a cutout L in the form of an infinitely thin circular slot of radius R located parallel to the half-space boundary at a depth H is examined (see the sketch). A cylindrical r, z, θ system of coordinates is selected with origin at the centre of the slot, where the z-axis is directed towards the free surface perpendicular to it. The half-space boundary is stress-free while the displacements equal zero at infinity.

We start from the complete system of equations of the axisymmetric theory of elasticity that describe the state of strain of a body /2/

$$(\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\,\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{q} = 0 \tag{1.1}$$

$$\sigma = \frac{\lambda}{r}\frac{\partial}{\partial r}(ru_1) + (\lambda + 2\mu)\frac{\partial u_2}{\partial z}, \quad \tau = \mu\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial r}\right)$$

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